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ON A VARIATIONAL DESCRIPTION OF MAGNETOHYDRODYNAMIC SYSTEMS INCLUDING IRREVERSIBLE PROCESSES

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SUMMARY

It is shown that Prigogine's evolution criterion describing dissipative hydrodynamic processes may be extended to magnetohydrodynamics. This is achieved by defining appropriately the generalized fluxes and forces. As illustrative example, Hartmann's flow is considered.

1. *Introduction*

In 1961, Prigogine [1] formulated an evolution criterion for dissipative processes. Starting from the well known property that the rate of entropy production in a volume Ω is of the form

$$P = \int J_\alpha X_\alpha d\Omega,^{(**)} \tag{1.1}$$

where J_α and X_α denote respectively the generalized fluxes and forces, Prigogine stated that, under time independant boundary conditions, the rate of entropy production can only decrease when the fluxes are maintained constant. This may be expressed as:

$$\frac{\partial_X P}{\partial t} = \int J_\alpha \frac{\partial X_\alpha}{\partial t} d\Omega \leq 0. \tag{1.2}$$

Later on, Prigogine and Glansdorff [2-5] extended this principle in order to include mechanical reversible processes as well. This is done by introducing a functional Φ, called generalized entropy production, whose time derivative is always negative when the system is evoluting and equal to zero when the steady state is reached:

$$\frac{\partial \Phi}{\partial t} = \int J'_\alpha \frac{\partial X'_\alpha}{\partial t} d\Omega \leq 0, \tag{1.3}$$

now, the generalized fluxes J'_α and forces X'_α contain not only irreversible but also reversible contributions.

One of the characteristics of Prigogine and Glansdorff's theory is the arbitrariness in the definition of fluxes and forces. Recently, Nihoul has shown [6] that by defining appropriately fluxes and forces, principle (1.2) is still fulfilled, even when the evolution of the considered system is governed by mechanical phenomena such as convection. Nihoul proceeds as follows: he starts from a result established by Eckart [7] that the rate of entropy production may still be written in the form (1.1) in the presence of con-

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(**) The sum convention on repeated indices is used throughout this paper.

vection. Nihoul calculates then the rate of entropy production within a volume Ω moving with the fluid and operates the separation between fluxes and forces by selecting as fluxes those factors whose divergence equals the material derivative of kinematical fundamental quantities (as velocity, total energy, etc...).

Nihoul's considerations are limited to incompressible fluids. In a previous note [8] we have suggested another separation procedure of the fluxes and forces which has the advantage of being more easily extended to compressible systems. On the one hand, we choose the fluxes so that their divergence expresses the rate of local variation (and not material variation as in Nihoul's work) of kinematical quantities. On the other hand, we define the forces from the expression of the rate of entropy production calculated within a fixed volume (instead within a moving volume as Nihoul did) and written in the form (1.1).

It must be noted that choosing a local formulation in order to define the fluxes is not new and can be found e.g. in references [9] and [10].

In the present work, we extend the analysis developed in reference [8] to conducting fluids. We show in particular that Prigogine's principle (1.2) remains valid for a magnetohydrodynamic flow, i.e. including dissipative processes as well as mechanical and magnetic ones.

After determining generalized fluxes and forces (§ 2), we show (§ 3) that, in the case of incompressible conducting fluids, the rate of entropy production, with constant fluxes, is either negative or equal to zero. We restrict our analysis to incompressible fluids for two reasons: firstly, it is the most usual situation in magnetohydrodynamics; secondly, the extension to compressible conducting fluids presents no difficulty.

In many cases, the evolution principle may be written in the form of a total differential. This is an important fact because it offers the possibility of using the techniques of the variational calculus to study the properties of the systems. In particular, the Euler-Lagrange equations of the variational problem are the conservation equations of the steady state. This property is illustrated by the example of Hartmann's flow of an incompressible fluid (§4).

2. *Generalized fluxes and forces.*

In order to determine the generalized fluxes, let us first write the fundamental equations of magnetohydrodynamics. As in most magnetohydrodynamic problems [10-12], displacement currents are neglected and Gauss' law is ommited. It is also assumed that the electrical conductivity σ, the magnetic permeability μ, the mass density ρ and the dynamic viscosity η are constant.

In addition, it is convenient instead of the usual notations $\underline{E}$, $\underline{B}$, $\underline{J}$ for the electrical field, the magnetic field and the current density, to use the local Alfvén notations $\underline{e}$, $\underline{b}$, $\underline{j}$ defined by.

$$\underline{b} = (\mu\rho)^{-\frac{1}{2}}\,\underline{B},$$

$$\underline{e} = (\mu\rho)^{-\frac{1}{2}}\,\underline{E},$$

$$\underline{j} = (\mu/\rho)^{\frac{1}{2}}\,\underline{J}.$$

For an incompressible fluid and using conventional electromagnetic M.K.S units, the fundamental equations of magnetohydrodynamics may then be written as:

$$\nabla\cdot\underline{v} = 0 \qquad \text{(conservation of mass equation)}, \tag{2.1}$$

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \rho(\underline{j}\wedge\underline{b}) + \eta\nabla^2\underline{v} \tag{2.2}$$

(Navier- Stokes equation),

$$\rho\frac{\partial}{\partial t}\left(\tfrac{1}{2}v^2+\epsilon+\tfrac{1}{2}b^2\right) = -\nabla.\left[\rho\underline{v}\left(\tfrac{1}{2}v^2+\epsilon\right)+\rho\underline{b}\Lambda(\underline{v}\Lambda\underline{b})+\pi.\underline{v}+\underline{q}+\lambda\rho\underline{b}\Lambda(\nabla\Lambda\underline{b})\right]$$

(Energy equation), (2.3)

$$\frac{\partial \underline{b}}{\partial t} = -\nabla\Lambda\underline{e} \qquad \text{(Faraday's law)}, \tag{2.4}$$

$$\underline{j} = \lambda(\underline{e}+\underline{v}\Lambda\underline{b}) \qquad \text{(Ohm's law)}, \tag{2.5}$$

$$\nabla\Lambda\underline{b} = \underline{j} \qquad \text{(Ampère's law)}, \tag{2.6}$$

$$\nabla.\underline{b} = 0 \tag{2.7}$$

with

$$\lambda = (\mu\sigma)^{-1}.$$

The quantity $\underline{v}$ is the fluid velocity, p the pressure, $\underline{q}$ the heat flux, ϵ the specific internal energy, π the pressure tensor with rectangular Cartesian components

$$\pi_{ij} = p\delta_{ij} - \eta\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right). \tag{2.8}$$

Let us now write equations (2.2), (2.3) and (2.4) in the form

$$\frac{\partial A_i}{\partial t} = -\frac{\partial}{\partial x_j}J_{ij} \tag{2.9}$$

where A_i represents either a kinematical or a magnetic quantity, either a combination of them and where J_{ij} represents the component of a generalized flux. Introducing (2.6) in Navier-Stokes equation (2.2), we get

$$\rho\frac{\partial v_i}{\partial t} = -\rho v_j\frac{\partial v_i}{\partial x_j} - \frac{\partial(p\delta_{ij})}{\partial x_i} - \frac{1}{2\rho}\frac{\partial(b^2\delta_{ij})}{\partial x_i} - \rho b_j\frac{\partial b_i}{\partial x_j} + \eta\frac{\partial}{\partial x_j}\left(\frac{\partial b_i}{\partial x_j}\right); \tag{2.10}$$

defining T_{ij} by

$$T_{ij} = \pi_{ij} + \rho v_i v_j - \rho(b_i b_j - \tfrac{1}{2}b^2\delta_{ij}), \tag{2.11}$$

equation (2.10) is of the form

$$\rho\frac{\partial v_i}{\partial t} = -\frac{\partial}{\partial x_j}T_{ij}$$

and it is clear that T_{ij} may be chosen as the components of the first flux.

On the other hand, by elimination of $\underline{e}$ and $\underline{j}$ between relations (2.4), (2.5) and (2.6), we obtain

$$\frac{\partial \underline{b}}{\partial t} = -\underline{v}\,.\,\nabla\,\underline{b} + \underline{b}.\nabla\,\underline{v} + \lambda\nabla^2\underline{b}, \tag{2.12}$$

which may also be expressed as

$$\rho\frac{\partial b_i}{\partial t} = -\frac{\partial}{\partial x_j}H_{ij}, \tag{2.13}$$

provided we define the flux H_{ij} as

$$H_{ij} = \rho b_i v_j - \rho v_i b_j - \rho\lambda \frac{\partial b_i}{\partial x_j}. \tag{2.14}$$

Finally, it appears directly from eq. (2.3) written in the form

$$\rho \frac{\partial}{\partial t}\left(\tfrac{1}{2}v^2+\epsilon+\tfrac{1}{2}b^2\right) = -\frac{\partial Q_j}{\partial x_j} \tag{2.15}$$

that the third flux is Q_j, given by

$$Q_j = q_j+\rho\epsilon v_j + \tfrac{1}{2}\rho v^2 v_j + \rho b^2 v_j - \rho v_i b_i v_j + \pi_{ij} v_i + \\ - \lambda\rho\left(\tfrac{1}{2}\frac{\partial b^2}{\partial x_j} - b_i\frac{\partial b_j}{\partial x_i}\right). \tag{2.16}$$

Our task is now to find the generalized forces corresponding respectively to the fluxes T_{ij}, H_{ij} and Q_j. Let us recall [8] that this is done by calculating the expression of the entropy production within a fixed volume containing the fluid.

For an incompressible fluid, the entropy variation per unit time in a fixed volume is given by

$$\frac{dS}{dt} = \int \rho\frac{\partial s}{\partial t}\, d\Omega = \int \theta^{-1}\rho\frac{\partial\epsilon}{\partial t}\, d\Omega, \tag{2.17}$$

where s denotes the specific entropy and θ the absolute temperature. Equation (2.17) may still be written

$$\frac{dS}{dt} = \int\left[\theta^{-1}\frac{\partial}{\partial t}(\rho\epsilon+\tfrac{1}{2}\rho v^2+\tfrac{1}{2}\rho b^2) - \theta^{-1}v_i\frac{\partial \rho v_i}{\partial t} - \theta^{-1}b_i\frac{\partial\rho b_i}{\partial t}\right] d\Omega \tag{2.18}$$

and, taking into account relations (2.11), (2.14) and (2.16),

$$\frac{dS}{dt} = \int\left(-\theta^{-1}\frac{\partial Q_j}{\partial x_j} + \theta^{-1}v_i\frac{\partial T_{ij}}{\partial x_j} + \theta^{-1}b_i\frac{\partial H_{ij}}{\partial x_j}\right) d\Omega. \tag{2.19}$$

By integration by parts, equation (2.19) is transformed into the sum of a volume integral and a surface integral. The latter corresponds to exchange of entropy with the outside whereas the former represents the rate of entropy production P within the volume and is given by

$$P = \int\left(\frac{\partial\theta^{-1}}{\partial x_j}Q_j - \frac{\partial\theta^{-1}v_i}{\partial x_j}T_{ij} - \frac{\partial\theta^{-1}b_i}{\partial x_j}H_{ij}\right) d\Omega. \tag{2.20}$$

From this expression, it seems rather natural to define the generalized forces corresponding to the fluxes Q_j, T_{ij} and H_{ij} respectively by

$$\frac{\partial\theta^{-1}}{\partial x_j}, \quad \frac{\partial\theta^{-1}v_i}{\partial x_j}, \quad \frac{\partial\theta^{-1}b_i}{\partial x_j}.$$

3. *Evolution criterion*

We now show that in the case of an incompressible conducting fluid, submitted to time independent boundary conditions, the rate of entropy production can only decrease when the fluxes J_α are maintained constant, namely

$$\frac{\partial_X P}{\partial t} = \int J_\alpha \frac{\partial X_\alpha}{\partial t} \, d\Omega \le 0, \tag{3.1}$$

the sign equality refers to the steady state.

Let us derive in expression (2.20) the forces with regard to the time, we obtain:

$$\frac{\partial_X P}{\partial t} = \int \left\{ \left[\frac{\partial}{\partial t} \left(\frac{\partial \theta^{-1}}{\partial x_j} \right) \right] Q_j - \left[\frac{\partial}{\partial t} \left(\frac{\partial \theta^{-1} v_i}{\partial x_j} \right) \right] T_{ij} - \left[\frac{\partial}{\partial t} \left(\frac{\partial \theta^{-1} b_i}{\partial x_j} \right) \right] H_{ij} \right\} d\Omega. \tag{3.2}$$

Let us permute spatial and time derivatives and integrate by parts; the boundary conditions being time independent, the surface integral vanishes and eq. (3.2) becomes

$$\frac{\partial_X P}{\partial t} = \int \left[\frac{\partial \theta^{-1}}{\partial t} \frac{\partial}{\partial t} \left(\rho \frac{v^2}{2} + \rho \epsilon + \rho \frac{b^2}{2} \right) - \frac{\partial \theta^{-1} v_i}{\partial t} \frac{\partial \rho v_i}{\partial t} - \frac{\partial \theta^{-1} b_i}{\partial t} \frac{\partial \rho b_i}{\partial t} \right] d\Omega. \tag{3.3}$$

Introducing now in this equation the thermodynamic relation

$$\frac{\partial \epsilon}{\partial t} = c \frac{\partial \theta}{\partial t} \quad (c > 0), \tag{3.4}$$

where c is the specific heat, we finally obtain:

$$\frac{\partial_X P}{\partial t} = - \int \left[\rho \theta^{-2} c \left(\frac{\partial \theta}{\partial t} \right)^2 + \rho \theta^{-1} \left(\frac{\partial v_i}{\partial t} \right)^2 + \rho \theta^{-1} \left(\frac{\partial b_i}{\partial t} \right)^2 \right] d\Omega \le 0, \tag{3.5}$$

which is undoubtedly a negative quantity, as requested.

Generally speaking, the quantity $\partial_X P / \partial t$ is not a total differential. However, as shown by Glansdorff and Prigogine [2-5], it is possible to write $\partial_X P / \partial t$ in the form of a total differential by linearizing around the stationary state, so that the techniques and the results of the calculus of variation can be used. In particular, the Euler-Lagrange equations of the variational problem are the steady state conservation equations; this property is illustrated in the following example.

4. *Hartmann's flow*

Let us consider the flow of an incompressible viscous fluid between two parallel solid planes when an uniform and constant external magnetic induction $\underline{b}_0$ is applied perpendicular to the planes.

It is natural to assume that the fluid velocity is everywhere in the same direction (we take it as the x_1 direction) and that the temperature is constant and uniform throughout the fluid. The components of the velocity are then

$$v_1, 0, 0. \tag{4.1}$$

In addition, the components of the magnetic field, i.e. the sum of the applied field and the induced field, are supposed to be given by

b_1, o, b_3,

with $b_3 = b_0$ = constant. (4.2)

Let us now calculate $\partial_X P/\partial t$ in the case of Hartmann's flow. This implies that we preliminarily determine the expressions of the fluxes H_{ij} and T_{ij}; it is not necessary to evaluate Q_j because the corresponding force $\partial\theta^{-1}/\partial x_j$ is equal to zero here. The components H_{ij} and T_{ij} which are needed to go further in the calculations are respectively:

$$H_{11} = -\rho\lambda \frac{\partial b_1}{\partial x_1}, \tag{4.3}$$

$$H_{13} = -\rho v_1 b_0 - \rho\lambda \frac{\partial b_1}{\partial x_3}, \tag{4.4}$$

$$T_{11} = p + \rho v_1^2 + \tfrac{1}{2}(b_0^2 - b_1^2) \tag{4.5}$$

and

$$T_{13} = -\eta \frac{\partial v_1}{\partial x_3} - \rho b_0 b_1. \tag{4.6}$$

Expression (3.2) of $\partial_X P/\partial t$ reduces now to:

$$\frac{\partial_X P}{\partial t} = -\theta^{-1}\int\left\{\left[\frac{\partial}{\partial t}\left(\frac{\partial v_1}{\partial x_1}\right)\right]T_{11} + \left[\frac{\partial}{\partial t}\left(\frac{\partial v_1}{\partial x_3}\right)\right]T_{13} + \left[\frac{\partial}{\partial t}\left(\frac{\partial b_1}{\partial x_1}\right)\right]H_{11} + \left[\frac{\partial}{\partial t}\left(\frac{\partial b_1}{\partial x_3}\right)\right]H_{13}\right\} d\Omega \leq 0 \tag{4.7}$$

and, with the help of eqs. (4.3) to (4.6):

$$\frac{\partial_X P}{\partial t} = \theta^{-1}\int\left\{\left[p + \rho v_1^2 + \tfrac{1}{2}(b_0^2 - b_1^2)\right]\frac{\partial}{\partial t}\left(\frac{\partial v_1}{\partial x_1}\right) + \tfrac{1}{2}\eta\frac{\partial}{\partial t}\left(\frac{\partial v_1}{\partial x_3}\right)^2 + \tfrac{1}{2}\rho\lambda\frac{\partial}{\partial t}\left(\frac{\partial b_1}{\partial x_1}\right)^2 + \tfrac{1}{2}\rho\lambda\frac{\partial}{\partial t}\left(\frac{\partial b_1}{\partial x_3}\right)^2 + \rho b_0\left[b_1\frac{\partial}{\partial t}\left(\frac{\partial v_1}{\partial x_3}\right) + v_1\frac{\partial}{\partial t}\left(\frac{\partial b_1}{\partial x_3}\right)\right]\right\} d\Omega \leq 0 \tag{4.8}$$

It is possible to write $\partial_X P/\partial t$ in the form of a total differential provided we restrict ourselves to fluctuations in the neighbourhood of the stationary state. Indeed, supposing the steady state characterized by the distribution

b_1^0, v_1^0

and linearizing around the steady state, eq. (4.8) becomes(*):

(*) It is not necessary to label ρ, η and λ with the supercript o because these quantities are constant.

$$\frac{\partial_X P}{\partial t} = \theta^{-1} \frac{\partial}{\partial t} \int \Big\{ - \Big[p^0 + \rho(v_1^0)^2 + \tfrac{1}{2} b_0^2 - \tfrac{1}{2}(b_1^0)^2 \Big] \frac{\partial v_1}{\partial x_1} + \tfrac{1}{2} \eta \left(\frac{\partial v_1}{\partial x_3}\right)^2$$

$$+ \tfrac{1}{2} \rho\lambda \left(\frac{\partial b_1}{\partial x_1}\right)^2 + \tfrac{1}{2} \rho \lambda \left(\frac{\partial b_1}{\partial x_3}\right)^2 + \rho b_0 \left(b_1^0 \frac{\partial v_1}{\partial x_3} + v_1^0 \frac{\partial b_1}{\partial x_3} \right) \Big\} d\Omega$$

$$= \theta^{-1} \frac{\partial}{\partial t} \int \Phi \, d\Omega \leq 0, \tag{4.9}$$

where the integrand Φ of (4.9) is given by:

$$\Phi = - \Big[p^0 + \rho(v_1^0)^2 + \tfrac{1}{2} b_0^2 - \tfrac{1}{2}(b_1^0)^2 \Big] \frac{\partial v_1}{\partial x_1} + \tfrac{1}{2} \eta \left(\frac{\partial v_1}{\partial x_3}\right)^2$$

$$+ \tfrac{1}{2} \rho \lambda \left(\frac{\partial b_1}{\partial x_1}\right)^2 + \tfrac{1}{2} \rho\lambda \left(\frac{\partial b_1}{\partial x_3}\right)^2 + \rho b_0 \left(b_1^0 \frac{\partial v_1}{\partial x_3} + v_1^0 \frac{\partial b_1}{\partial x_3} \right) \tag{4.10}$$

It is important to note that functional Φ depends on both the steady state variables b_1^0, v_1^0 and the usual variables b_1, v_1 and takes his minimum values in the stationary state.

The Euler-Lagrange equations corresponding to (4.9) and (4.10) namely

$$\frac{\partial}{\partial x_j} \frac{\partial \Phi}{\partial \left(\frac{\partial v_1}{\partial x_j}\right)} - \frac{\partial \Phi}{\partial v_1} = 0, \tag{4.11}$$

$$\frac{\partial}{\partial x_j} \frac{\partial \Phi}{\partial \left(\frac{\partial b_1}{\partial x_j}\right)} - \frac{\partial \Phi}{\partial b_1} = 0, \tag{4.12}$$

associated with the subsidiary conditions

$$v_1 = v_1^0, \quad b_1 = b_1^0 \tag{4.13}$$

and the fact that b_o is constant and uniform give:

$$- \frac{\partial}{\partial x_1} \Big[p^0 + \rho(v_1^0)^2 - \tfrac{1}{2}(b_1^0)^2 \Big] + \frac{\partial}{\partial x_3}\left(\eta \frac{\partial v_1^0}{\partial x_3} + \rho b_0 b_1^0 \right) = 0, \tag{4.14}$$

$$\rho\lambda \frac{\partial}{\partial x_1} \left(\frac{\partial b_1^0}{\partial x_1}\right) + \rho \frac{\partial}{\partial x_3} \left(\lambda \frac{\partial b_1^0}{\partial x_3} + b_0 v_1^0 \right) = 0. \tag{4.15}$$

Moreover, the magnetohydrodynamic relations (2.1) and (2.7)

$$\nabla \cdot \underline{v} = 0 \text{ and } \nabla \cdot \underline{b} = 0$$

imply that

$$\frac{\partial v_1^0}{\partial x_1} = 0 \text{ and } \frac{\partial b_1^0}{\partial x_1} = 0$$

so that the expressions (4.14) and (4.15) may finally be written in the form

$$\eta \frac{\partial^2 v_1^0}{\partial x_3^2} + \rho b_0 \frac{\partial b_1^0}{\partial x_3} = \frac{\partial p^0}{\partial x_1} \qquad (4.16)$$

$$\lambda \frac{\partial^2 b_1^0}{\partial x_3^2} + b_0 \frac{\partial v_1^0}{\partial x_3} = 0 \qquad (4.17)$$

which are the well-known steady state equations of Hartmann's flow (see for instance ref. 10).

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